REDUCED COMPLEXITY AOR SCHEME FOR THE SOLUTION OF TWO DIMENSIONAL POISSON PROBLEMS

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Abstract: Development of a numerical system for the solution of partial differential equation is very crucial since it can simulate the behavior of many scientific phenomena. The developed system can be used by researchers, educators and student for research and education purposes. The system was developed using three accelerated over relaxation (AOR) schemes to solve two dimensional Poisson's equations. In addition, the results generated by the system can be exported to Microsoft Excel for further analysis. This system was developed using C # programming languages and ASP.NET. Microsoft SQL Server 2005 is used as a database to store data. In this research, the successive over relaxation (SOR) method was used as the control of comparison. From the research, the Quarter Sweep Accelerative Over Relaxation (QSAOR) gives the best performance as compared to others.

Keywords: Accelerative over relaxation, quarter sweep accelerative over relaxation, rotated accelerative over relaxation, Poisson equation.

Introduction

In the era of information and communication technology, information technology plays an important role in everyday's activity such as in industrial work, research and system development. Rapid progress in microelectronics development has resulted in various information technology facilities such as personal computers, telecommunications, multimedia and many others. This technology allows information sharing, simulations and many other activities processed or managed via computers or even used to solve complex problems that can be mimicked by mathematical formulations.

Elliptic partial differential equations (PDEs) are very important in most scientific and engineering simulations activities. Elliptic PDE can be used to simulate various problems arise in many branches of science and engineering, such as steady temperature distribution in a two-dimensional field of constant conductivity, distribution of electric or magnetic potential, temperature distribution, travelling wave in guiding conductors, electrostatics, and groundwater flow. For instance, the distribution of the electrostatic potential can be determined by solving Poisson equation, if there is a charge density in problem domain, otherwise if the simulation domain is free from electric charge, the governing equation is known as Laplace equation.

Numerical solutions have been used by researchers to solve Elliptic problem [1-6] for so many years. Various approaches have been proposed by researchers such as the Explicit Decoupled

Group [1, 4, 6, 7, 8] and Modified Explicit Group [2, 3] to solve the elliptic problem. The Explicit Decoupled Group utilized the rotated scheme which only solved half of the solution domain, while the Modified Explicit Group utilized the *2h* mesh size and solved only a quarter of the solution domain. This approach has been extended to develop other methods such as the Half Sweep Iterative Alternating Decomposition Explicit scheme [9], Quarter Sweep Arithmetic Mean to solve fourth order parabolic equation [10] and Quarter Sweep Iterative Alternating Decomposition Explicit scheme to solve diffusion equation [11]. The concept has been further extended to produce complexity reduction approach methods such as the High Speed Low Order Finite Difference Time Domain [12], High Speed High Order Finite Difference Time Domain [13, 14]. Some literature on this complexity reduction approach can be read in [15]. The extension to parallel algorithm has been done to further increase the speed of the High Speed Low Order Finite Difference Time Domain [16]. These methods have shown a very good performance, primarily on the computational aspect to solve free space wave propagation problem.

In this paper, we will discuss the development of various fast relaxation schemes for solving two dimensional Poisson equations given by Eq. (1) with Dirichlet boundary conditions given by Eq. (2).

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega = [a,b]\times[a,b] \qquad (1)$$

$$U(x,y) = g(x,y), \quad (x,y) \in d\Omega \qquad (2)$$

Eq. (1) is discretized using some finite difference scheme based on over relaxation approaches.

Materials and Methods

Consider a linear system of equation given by $A\tilde{x} = \tilde{b}$. A system of linear equations for Eq. (1) can be constructed via finite difference approximation. If the coefficient matrix, A is decomposed into block diagonal, D, lower triangular, L, and upper triangular, U, using relation given in (3).

$$A = D - L - U \qquad (3)$$

Hadjidimos [17] utilized Eq. (3) to propose a method called Accelerated Over Relaxation (AOR). The method is defined as Eq. (4).

$$u^{(k+1)} = L_{r,w}u^{(k)} + w(D - rL)^{-1}b,$$
$$L_{r,w} = (I - rD^{-1}L)^{-1}\left[(1-w)I + (w-r)L + wD^{-1}U\right] \quad (4)$$

Eq. (4) can be re-written as

$$(D - rL)\tilde{u}^{(k+1)} = (w-r)L\tilde{u}^{(k)} + wU\tilde{u}^{(k)} + w\tilde{b} + (1-w)D\tilde{u}^{(k)}. \quad (5)$$

Re-arranging Eq. (5), we arrive at

$$D\tilde{u}^{(k+1)} = rL\tilde{u}^{(k+1)} + wU\tilde{u}^{(k)} + wb + (1-w)D\tilde{u}^{(k)}. \quad (6)$$

Solving problem (1) using the finite difference approximation in a rectangular grid in (x, y) plane with equal grid spacing in both x and y directions with $x_i = ih,\ y_j = jh(i, j = 0,1,2,\cdots,N)$. In this paper, the point $u(x_i, y_j)$ is denoted by u_{ij}. Since in this paper an equal square mesh is used, the mesh size, $h = \frac{1}{N}$. By truncating the approximation of problem (1) at $O(h^2)$, we obtain an approximation equation which is known as the five-point difference approximation and given as Eq. (7).

$$u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j} - 4u_{ij} = h^2 f_{ij}. \quad (7)$$

The Successive Over Relaxation (SOR) iterative scheme for Eq. (7) is gathered by introducing a weighted parameter, w and can be written as Eq. (8).

$$u_{i,j}^{(k+1)} = w\left(\frac{u_{i-1,j}^{(k+1)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(+1)} + u_{i,j+1}^{(k)} - h^2 f_{i,j}}{4}\right) + (1-w)u_{i,j}^{(k)}. \quad (8)$$

The AOR iterative scheme for Eq. (7) can be constructed by introducing a parameter, r and replace $u_{i-1,j}^{(k+1)}$ with $u_{i-1,j}^{(k)}$ and $u_{i,j-1}^{(k+1)}$ with $u_{i,j-1}^{(k)}$, and adding $\frac{r\left(u_{i-1,j}^{(k+1)} - u_{i-1,j}^{(k)}\right)}{4}$ and $\frac{r\left(u_{i,j-1}^{(k+1)} - u_{i,j-1}^{(k)}\right)}{4}$ to the SOR scheme and shown as Eq. (9).

$$u_{i,j}^{(k+1)} = r\left(\frac{u_{i-1,j}^{(k+1)} + u_{i-1,j}^{(k)} + u_{i,j-1}^{(k+1)} + u_{i,j-1}^{(k)}}{4}\right)$$
$$+ w\left(\frac{u_{i-1,j}^{(k)} + u_{i+1}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)} - h^2 f_{i,j}}{4}\right) + (1-w)u_{i,j}^{(k)}. \quad (9)$$

Another variant of AOR schemes can be constructed by rotating the x axis and the y axis by 45^0 clockwise [1, 7, 8, 9]. The location of nodes used in Eq. (7) transform as below.

$$i,j\pm 1 \rightarrow i\pm 1, j\pm 1,$$
$$i\pm 1, j \rightarrow i\pm 1, j\mp 1,$$
$$\Delta x, \Delta y \rightarrow \sqrt{(\Delta x)^{2+}(\Delta x)^2} = \sqrt{2}h,\ \Delta x = \Delta y,$$

Thus, Eq. (7) transform into Eq. (10).

$$u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} - 4u_{ij} = 2h^2 f_{ij}. \quad (10)$$

By following the SOR approach, we arrive at Eq. (11).

$$u_{i,j}^{(k+1)} = w\left(\frac{u_{i-1,j-1}^{(k+1)} + u_{i+1,j+1}^{(k)} + u_{i+1,j-1}^{(k+1)} + u_{i-1,j+1}^{(k)} - 2h^2 f_{i,j}}{4}\right) + (1-w)u_{i,j}^{(k)}. \quad (11)$$

The Rotated Accelerated Over Relaxation (RAOR) iterative scheme for Eq. (10) can be developed by introducing a parameter, r and replace $u_{i-1,j-1}^{(k+1)}$ with $u_{i-1,j-1}^{(k)}$ and $u_{i+1,j-1}^{(k+1)}$ with $u_{i+1,j-1}^{(k)}$, and adding $\frac{r\left(u_{i-1,j-1}^{(k+1)} - u_{i-1,j-1}^{(k)}\right)}{4}$ and $\frac{r\left(u_{i+1,j-1}^{(k+1)} - u_{+1i,j-1}^{(k)}\right)}{4}$ in Eq. (11). Thus gives

$$\begin{aligned} u_{i,j}^{(k+1)} &= r\left(\frac{u_{i-1,j-1}^{(k+1)} + u_{i-1,j-1}^{(k)} + u_{i+1,j-1}^{(k+1)} + u_{i+1,j-1}^{(k)}}{4}\right) \\ &+ w\left(\frac{u_{i-1,j-1}^{(k)} + u_{i+1,j-1}^{(k)} + u_{i-1,j+1}^{(k)} + u_{i+1,j+1}^{(k)} - 2h^2 f_{i,j}}{4}\right) + (1-w)u_{i,j}^{(k)} \end{aligned} . \quad (12)$$

The third AOR scheme variant can be gathered as follows.

$$u_{i,j+2} + u_{i,j-2} + u_{i+2,j} + u_{i-2,j} - 4u_{ij} = 4h^2 f_{ij}. \quad (13)$$

By introducing a weighted parameter w, into a SOR-like approach, we arrive at Eq. (14).

$$u_{i,j}^{(k+1)} = w\left(\frac{u_{i-2,j}^{(k+1)} + u_{i+2,j}^{(k)} + u_{i,j-2}^{(+1)} + u_{i,j+2}^{(k)} - 4h^2 f_{i,j}}{4}\right) + (1-w)u_{i,j}^{(k)}. \quad (14)$$

By introducing a new weighted parameter, r and replacing $u_{i-2,j}^{(k+1)}$ with $u_{i-2,j}^{(k)}$ and $u_{i,j-2}^{(k+1)}$ with $u_{i,j-2}^{(k)}$, and adding $\frac{r\left(u_{i-2,j}^{(k+1)} - u_{i-2,j}^{(k)}\right)}{4}$ and $\frac{r\left(u_{i,j-2}^{(k+1)} - u_{i,j-2}^{(k)}\right)}{4}$ in Eq. (14), we arrive at Eq. (15). Eq. (15) will be known as Quarter Sweep Accelerated Over Relaxation (QSAOR) method.

$$u_{i,j}^{(k+1)} = r\left(\frac{u_{i-2,j}^{(k+1)} + u_{i-2,j}^{(k)} + u_{i,j-2}^{(k+1)} + u_{i,j-2}^{(k)}}{4}\right) + w\left(\frac{u_{i-2,j}^{(k)} + u_{i+2,j}^{(k)} + u_{i,j-2}^{(k)} + u_{i,j+2}^{(k)} - 4h^2 f_{i,j}}{4}\right) + (1-w)u_{i,j}^{(k)} . \quad (15)$$

Results and Discussion

In order to show the performance of the Accelerative over relaxation (AOR), Rotated accelerative over relaxation (RAOR) and Quarter sweep accelerative over relaxation (QSAOR) methods, we conducted an experiment and recorded the performance parameter such as accuracy, number of iteration and computational time and compared it to the famous Successive Over Relaxation (SOR) method.

In this paper, we consider the problem of two dimensional Poisson problems in Eq. (15) and its boundaries condition in Eq. (16).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) = (x^2 + y^2)e^{xy}, \quad (15)$$

in R, (x,y) ∈ (0, 1)×(0, 1), with boundary conditions as in Eq. (16).

$$u(x,0) = u(0,y) = 1, u(x,1) = e^x, u(1,y) = e^y . \quad (16)$$

The exact solution for Eqs. (15) and (16) is given in (17).

$$u(x,y) = e^{xy} . \quad (17)$$

The convergence criteria used in this research was,

$$\text{maks}\left\{\left|u_{i,j}^{(k+1)} - u_{i,j}^{(k)}\right|\right\} < 10^{-6}, \quad (18)$$

as the stopping criterion.

In every experiment, the r values chosen are close to the optimum value of w . The selected parameter w should give the minimum number of iteration. Every experiment was conducted three times and the average are recorded in Table 1.

Table 1. Comparison of computational time for compared methods

Methods	N	t_1	t_2	t_3	Average (seconds)

SOR	16	0.015625	0	0.046875	0.020833
	2916	0.5625	0.484375	0.453125	0.5
	10816	1.875	1.78125	1.90625	1.854167
AOR	16	0	0.046875	0	0.015625
	2916	0.328125	0.34375	0.265625	0.3125
	10816	1.859375	1.796875	1.8125	1.822917
RAOR	16	0.015625	0.046875	0	0.020833
	2916	0.203125	0.15625	0.140625	0.166667
	10816	0.765625	0.8125	0.78125	0.786458
QSAOR	16	0	0	0	0
	2916	0.0625	0.140625	0.125	0.109375
	10816	0.453125	0.359375	0.4375	0.416667

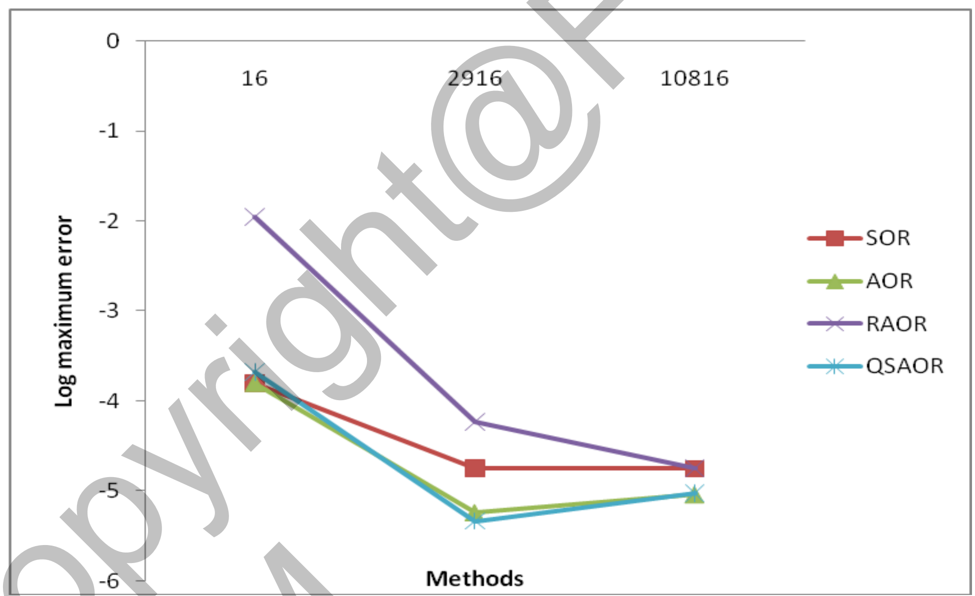

Figure 1. Accuray of compared methods

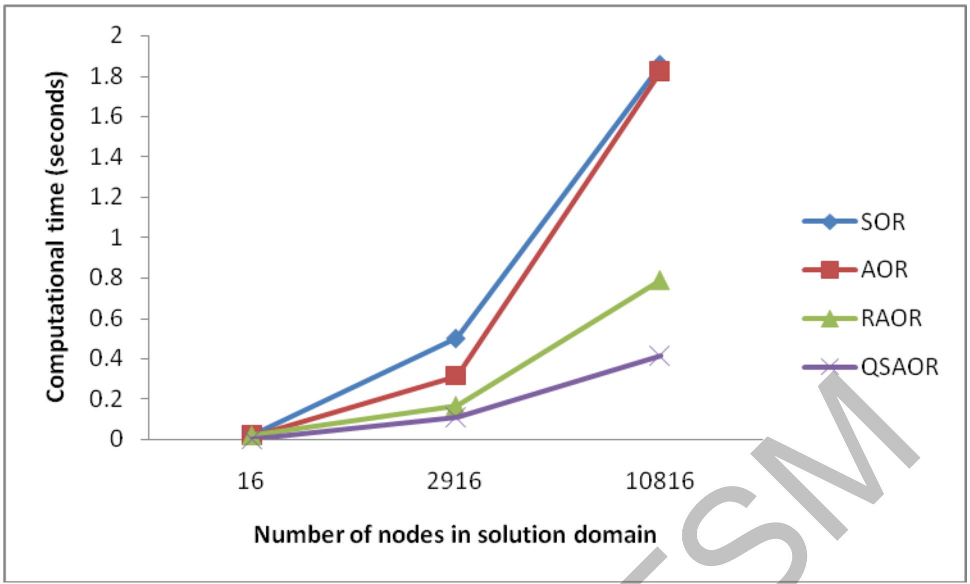

Figure 2. Comparison of computational time

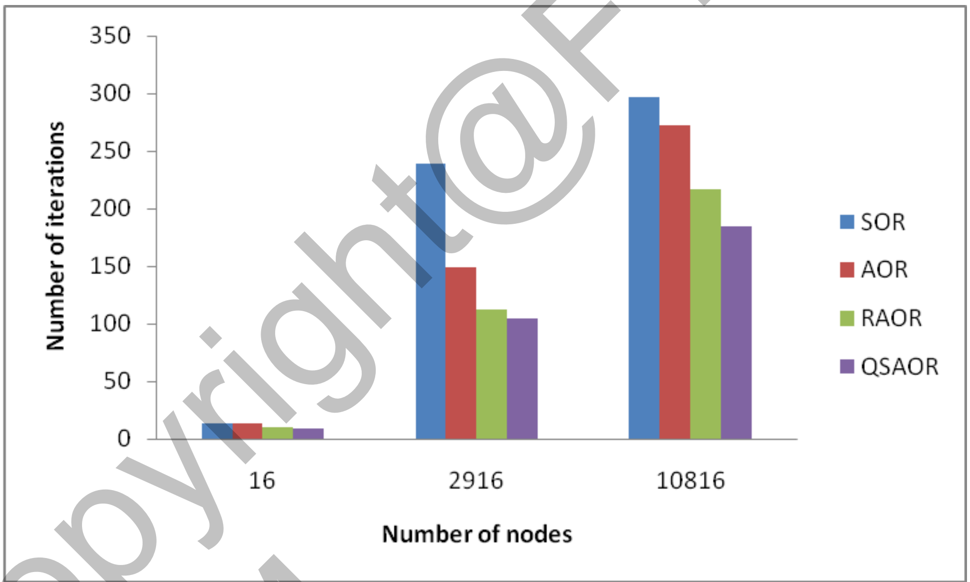

Figure 3. Comparison of iterations number

The numerical experiment results (accuracy, computational time and number of iterations) were given in Figs. 1, 2 and 3. From the table, it clearly shows that using more node points to discretize the solution domain will result in higher accuracy for all numerical methods.

For AOR, the improvement of accuracy gathered by discretizing the solution domain using 2916 node points is 1.37 more accurate compared to 16 nodes. However, by increasing the node points to 10816 does not significantly improved the accuracy as compared to 2916 nodes. For HSAOR, the improvement is even better, since solving the problem using 2916 node points is 2.16 more accurate compared to 16 nodes. While for QSAOR, solving using 2916 nodes is 1.45 more accurate than solving using 16 nodes. Increasing the node points to 10816 results in almost the same accuracy compared to 2916 nodes. Comparing AOR schemes to SOR, for n=10816, AOR and QSAOR have better accuracy as compared to SOR. However, RAOR is less accurate as compared to SOR. The situation is different for n=2916. For n=2916, QAOR is better than AOR, SOR and RAOR. However, RAOR still have the lowest accuracy amongst all compared methods.

In comparing computational time, all AOR, RAOR and QSAOR methods show that more computational time is needed for a higher number of node points. RAOR needs lower number of iteration than AOR and QSAOR needs lower number of iteration as compared to RAOR. In theory, RAOR have half less complexity as compared to AOR and QSAOR have three quarter less complexity than AOR. Both complexity and number of iteration is needed to explain the result of all AOR schemes that we gathered in the experiment. RAOR is almost 1.35-1.99 times faster than AOR (from theory, RAOR have half less complexity than AOR) while QSAOR is almost 3.45-3.57 times faster than AOR (from theory, QSAOR have three quarter less complexity than AOR). Comparing all the AOR schemes to SOR, AOR are 1.01-1.6 times faster than SOR. This may be the cause of AOR iteration number also lower than SOR by 1.07-1.60 times. RAOR are 1.00-3.00 times faster than SOR, since it iteration number are 1.4-2.13 times lower than SOR and the RAOR only solve 50% of solution domain iteratively. Meanwhile, QSAOR are 4.44-4.57 times faster than SOR method since QSAOR iteration number are 1.55-2.28 times lower and the method only solve 25% of solution domain iteratively.

Conclusions

Three over relaxation methods, which are AOR, RAOR and QSAOR methods have been demonstrated in this paper, while SOR method has been used for control of comparison. All methods show to converge to the exact solution. Using the concept of quarter sweep, QSAOR have successfully reduced the complexity of AOR method and resulted in faster computing time. However, by applying the rotated strategy, a method call RAOR is created and have successfully reduced the computational time by half. In addition to that, this method needs less number of iteration than SOR method.

As a conclusion, the most optimal method is the QSAOR method as compared to SOR, AOR and RAOR methods since it needs the least number of iteration and computing time. It also ranked as having the second accurate result as compared to SOR, AOR and RAOR method for n=10816, and the most accurate method for n=2916. In the future, we are planning to implement fuzzy approach to solve the problem and to develop new fuzzy methods following the approach in [18-21].

Acknowledgements

We acknowledge the financial support from GGP-2017-023 grant.

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