Sensitivity Analysis of the Lotka-Volterra Type Model for GDP and FDI Dynamics Using the RK4 Method

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Abstract

In this study, we analyse a modified Lotka-Volterra system designed to model the interaction between Gross Domestic Product (GDP) and Foreign Direct Investment (FDI). The system of differential equations incorporates nonlinear terms that reflect the interdependent growth rates of GDP and FDI, akin to predator-prey dynamics. We employ the Runge-Kutta 4th Order (RK4) method to numerically solve this system and perform a sensitivity analysis on key parameters. The results demonstrate how variations in these parameters influence the GDP and FDI dynamics, highlighting the model's sensitivity and implications for economic policy.

Keywords: Lotka-Volterra, GDP, FDI, RK4 method, sensitivity analysis, economic modelling, nonlinear dynamics

Introduction

The Lotka-Volterra equations are well known in ecological modelling, specifically for describing predator-prey interactions. However, variations of the Lotka-Volterra system have been applied in other domains such as economics, where analogous interactions exist between key variables like Gross Domestic Product (GDP) and Foreign Direct Investment (FDI). GDP represents the total economic output, while FDI captures the investment inflows from foreign entities. The interaction between these two variables is complex, as GDP can both attract and benefit from FDI, while FDI can either foster growth or be constrained by economic performance.

In this study, we examine a system of equations inspired by the Lotka-Volterra model to describe the dynamics between GDP and FDI. The equations are as in Eq. (1).

$$\left.\begin{aligned} \frac{dGDP}{dt} &= -12.65GDP + 3.36GDP^2 - 2.26GDP * FDI \\ \frac{dFDI}{dt} &= -44.65FDI - 6.17FDI^2 + 10.83GDP * FDI \end{aligned}\right\} \quad \text{Eq. (1)}$$

This model introduces nonlinear terms and interaction effects that reflect the interdependent dynamics between GDP and FDI. To understand the robustness of the model, we perform a sensitivity analysis using the Runge-Kutta 4th Order (RK4) method to solve the system numerically and assess the impact of small parameter variations on system behaviour.

Sensitivity analysis is a crucial technique in the field of mathematical modelling and simulation, used to determine how different values of an input parameter affect a particular output variable under a given set of assumptions. This method is essential for understanding the robustness and reliability of models, especially in scenarios where precise data may be lacking or where models are used to predict outcomes in complex systems. Sensitivity analysis helps in identifying which parameters are most influential on the output, thereby guiding researchers and decision-makers in focusing their efforts on the most critical variables (Thabane et al. 2013)

One of the primary applications of sensitivity analysis is in the field of environmental modelling, where it is used to assess the impact of various environmental factors on ecosystem dynamics. For instance, in climate change models, sensitivity analysis can help determine how changes in greenhouse gas concentrations, temperature, and precipitation patterns influence climate predictions. This information is vital for developing effective mitigation and adaptation strategies. Similarly, in hydrological models, sensitivity analysis can identify the key factors affecting water flow and quality, aiding in the management of water resources (Looss & Saltelli 2016).

In the realm of economics and finance, sensitivity analysis is employed to evaluate the stability of economic models and forecasts. By varying key economic indicators such as interest rates, inflation rates, and employment levels, analysts can assess the potential risks and uncertainties associated with economic predictions. Sensitivity analysis also plays a significant role in cost-benefit analysis, helping to identify the most cost-effective strategies under different economic scenarios (Helton & Davis 2003; Saltelli et at. 2008).

Methodology

GDP-FDI Dynamic Model

The system of equations captures the interactions between GDP and FDI over time. The first equation governs the time evolution of GDP, considering both positive growth effects (represented by the GDP^2 term) and negative effects due to interaction with FDI. The second equation governs the time evolution of FDI, with terms for natural decay and growth driven by interactions with GDP.

Numerical Solution Using RK4

The RK4 method is a common numerical technique for solving differential equations due to its balance between accuracy and computational efficiency. For the system given by:

$$\frac{dGDP}{dt} = f_1(GDP, FDI)$$

$$\frac{dFDI}{dt} = f_2(GDP, FDI)$$

The RK4 method computes the solution incrementally, updating the variables GDP and FDI at each time step t_n. The iterative steps are as follows:

$$k_{1,GDP} = hf_1(GDP_n, FDI_n), \qquad k_{1,FDI} = hf_2(GDP_n, FDI_n)$$

$$k_{2,GDP} = hf_1\left(GDP_n + \frac{k_{1,GDP}}{2}, FDI_n + \frac{k_{1,FDI}}{2}\right)$$

$$k_{3,GDP} = hf_1\left(GDP_n + \frac{k_{2,GDP}}{2}, FDI_n + \frac{k_{2,FDI}}{2}\right)$$

$$k_{4,GDP} = hf_1\left(GDP_n + \frac{k_{3,GDP}}{2}, FDI_n + \frac{k_{3,FDI}}{2}\right)$$

$$GDP_{n+1} = GDP_n + \frac{1}{6}(k_{1,GDP} + 2k_{2,GDP} + 2k_{3,GDP} + k_{4,GDP})$$

Similar steps are used for FDI. The method allows for accurate tracking of GDP and FDI dynamics over time.

Sensitivity Analysis

To conduct a sensitivity analysis for the given system of differential equations, we apply One-at-a-time (OAT) analysis approach (Yu et al. 2019). This process involves determining how sensitive the system's outcomes (GDP and FDI) are to changes in the model's parameters.

System of Equations:

$$\left.\begin{aligned}\frac{dGDP}{dt} &= -12.65GDP + 3.36GDP^2 - 2.26GDP * FDI \\ \frac{dFDI}{dt} &= -44.65FDI - 6.17FDI^2 + 10.83GDP * FDI\end{aligned}\right\}$$

Step-by-Step Sensitivity Analysis

Step 1: Identify the Parameters

The parameters in your model are:

$a = -12.65$ coefficient of GDP in the first equation

$b = 3.36$ coefficient of GDP^2 in the first equation

$c = -2.26$ coefficient of $GDP * FDI$ in the first equation

$d = -44.65$ coefficient of FDI in the second equation)

$e = -6.17$ coefficient of FDI^2 in the second equation

$f = 10.83$ coefficient of $GDP * FDI$ in the second equation

Step 2: Define Baseline Values

Define the initial conditions and baseline values for the parameters and variables (GDP and FDI). For example:

Initial $GDP = GDP_0 = 6.299$

Initial $FDI = FDI_0 = 0.512$

Step 3: Solve the System of Equations (Baseline Case)

Using numerical methods such as the Runge-Kutta 4th Order Method (RK4), solve the system of differential equations with the baseline parameter values. Record the outputs (GDP and FDI) over time.

Step 4: Perturb the Parameters and solve the problem.

To analyze sensitivity, slightly perturb each parameter individually. For each parameter perturbation, solve the system again using the same initial conditions and numerical methods. Compare the results of the perturbed system (Table 2) with the baseline case (Table 1). Calculate the relative changes in the GDP and FDI values to determine how sensitive the system is to each parameter.

Table 1: Baseline case accuracy

Coefficient	Optimum	
Model	GDP	FDI
RMSE	0.0065	0.7254
MAPE	0.00034	4.38494

Step 5: Calculate Sensitivity Indices

A common way to quantify sensitivity is by calculating sensitivity indices. For each parameter, the sensitivity index S can be defined as:

$$S = \frac{\%\ Decrease\ in\ accuracy(RMSE)}{\%\ Change\ in\ Parameter}$$

Table 2: Accuracy of LV model for Increase and decrease in coefficients

Coefficient	a -6%		b -4.5%		c -4.5%	
Model	GDP	FDI	GDP	FDI	GDP	FDI
RMSE	0.0065	1.4661	0.0065	1.3448	0.0065	1.4931
MAPE	0.00036	17.9118	0.00036	15.071	0.00036	18.57707
S	0	8.42030	0	7.67648	0	8.569419
Coefficient	d -0.6%		e -0.6%		f -3%	
Model	GDP	FDI	GDP	FDI	GDP	FDI
RMSE	0.0065	1.320	0.0065	1.4717	0.0065	1.4763
MAPE	0.00036	14.529	0.00036	18.05036	0.00036	18.1614
S	0	75.0757576	0	84.516772	0	16.9545485
Coefficient	a +0.6%		b +3%		c +1.1%	
Model	GDP	FDI	GDP	FDI	GDP	FDI
RMSE	0.0065	1.4661	0.0065	1.3448	0.0065	1.4931
MAPE	0.00036	17.9118	0.00036	15.071	0.00036	18.57707
S	0	78.8671024	0	16.6655178	0	44.7449645
Coefficient	d +5%		e +7%		f +0.4%	
Model	GDP	FDI	GDP	FDI	GDP	FDI
RMSE	0.0065	1.403	0.0065	1.4821	0.0065	1.323
MAPE	0.00036	16.411	0.00036	18.3065	0.00036	14.585
S	0	9.6593015	0	7.29370488	0	112.92517

We found that, percentage of changes allowed to maintain at least good accuracy for coefficients are difference. For $a = (-6\%, +0.6\%), b = (-4.5\%, +3\%), c = (-4.5\%, 1.1\%), d = (-0.6\%, +5\%), e = (-0.6\%, +7\%)$ and $f = (-3\%, +0.4\%)$.

Sensitivity Analysis of Model Coefficients

The robustness of the model is evaluated by examining the permissible variations in the coefficients (a, b, c, d, e, f) without significantly compromising its accuracy. Each coefficient exhibits a specific range within which it can vary while maintaining reliable results. For instance, coefficient a can decrease by up to 6% or increase by 0.6% without substantially affecting the model's performance, whereas coefficient e is more flexible, allowing an increase of up to 7% and a decrease of 0.6%. This indicates that some coefficients are more sensitive to changes than others, and the model's performance is contingent upon maintaining these coefficients within their allowable ranges.

Coefficient a: A reduction of 6% in coefficient a result in a sensitivity index ($S = 8.42$), whereas an increase of 0.6% yields a sensitivity index ($S = 78.87$). This demonstrates that the model is highly sensitive to increases in coefficient a compared to decreases. The significant disparity in sensitivity indices suggests that even minor upward adjustments in a can substantially impact the model's behaviour, necessitating careful control of any increases.

Coefficient b: A reduction of 4.5% in coefficient b results in a sensitivity index ($S = 10.24$), while an increase of 3% yields a sensitivity index ($S = 16.67$). The model exhibits moderate sensitivity to both increases and decreases in b, with a slightly greater impact observed for increases. This indicates that changes in b, whether upward or downward, affect the model to a similar extent, though increases have a somewhat stronger effect.

Coefficient c: A reduction of 4.5% in coefficient c results in a sensitivity index ($S = 11.43$), whereas an increase of 1.1% yields a sensitivity index ($S = 44.74$). The model is significantly more sensitive to increases in c than to decreases. This suggests that upward changes in c must be managed carefully, as they lead to much greater shifts in model performance compared to downward adjustments.

Coefficient d: A reduction of 0.6% in coefficient d results in a sensitivity index ($S = 75.08$), while an increase of 5% yields a sensitivity index ($S = 9.66$). The model is highly sensitive to decreases in d but less sensitive to increases. This indicates that even small reductions in d have a significant impact on the model's performance, whereas larger increases are more tolerable.

Coefficient e: A reduction of 0.6% in coefficient e results in a sensitivity index ($S = 84.52$), whereas an increase of 7% yields a sensitivity index ($S = 7.29$). The model is extremely sensitive to decreases in e but shows much less sensitivity to increases. This suggests that the model is particularly vulnerable to reductions in e, while increases are far less disruptive.

Coefficient f: A reduction of 3% in coefficient f results in a sensitivity index ($S = 16.95$), whereas an increase of 0.4% yields a sensitivity index ($S = 112.93$). The model is much more sensitive to increases in f than to decreases. This indicates that even small upward adjustments in f can dramatically affect the model's performance, necessitating great caution in managing increases.

Overall Sensitivity Analysis:

The model's performance is highly variable depending on the direction and magnitude of changes in each coefficient. Coefficients a, c and f exhibit greater sensitivity to increases than decreases, suggesting that upward adjustments in these coefficients must be carefully controlled. Conversely, coefficients d and e show higher sensitivity to decreases, indicating that downward changes in these coefficients should be avoided. Coefficient b demonstrates moderate sensitivity to both increases and decreases, with a slightly stronger impact from upward adjustments.

In summary, the model is more sensitive to upward changes in coefficients a, c and f, while it is more sensitive to downward changes in coefficients d and e. Coefficient b exhibits a more balanced sensitivity. Careful tuning of each coefficient within its allowable range is essential to maintain model accuracy and performance.

Model Robustness:

The model is considered robust as it can tolerate certain variations in the coefficients while maintaining good accuracy. Coefficients like d and e are less sensitive to increases, allowing for more flexibility in adjustments without losing accuracy. However, coefficients such as a and f require more cautious changes, as even small shifts outside their limits could affect the model's predictions. This balance of sensitivity in each coefficient demonstrates the model's ability to handle slight changes while still performing effectively.

Strengths and Weaknesses:

Strengths:

- **Balanced Sensitivity:** The model shows robustness, particularly with coefficients like b, which exhibit moderate sensitivity to both increases and decreases. This suggests that the model can tolerate some changes without severely affecting its performance.
- **Clear Sensitivity Patterns:** The distinct sensitivity patterns across different coefficients provide a clear understanding of where adjustments can be made with caution, aiding in fine-tuning the model in a controlled manner.

Weaknesses:

- **High Sensitivity in Key Areas:** The model shows extreme sensitivity in certain coefficients, such as f, where even a small increase leads to a significant drop in performance. Similarly, small decreases in d and e severely impact accuracy. These high sensitivity levels suggest that the model may not handle changes or noise in data well, potentially reducing its reliability in practical applications.
- **Limited Flexibility:** The tight ranges for many coefficients (especially a, c and f) imply that the model lacks flexibility, making it challenging to adapt to different data sets or scenarios without sacrificing accuracy.

Implications for Economic Policy

The sensitivity analysis demonstrates that while interaction between GDP and FDI is essential for economic growth, overly strong interactions can result in amplified oscillations and instability. Policies that aim to balance GDP growth and FDI inflows should focus on moderating the interaction terms to prevent excessive economic fluctuations.

Conclusion

This study applied the RK4 method to solve a Lotka-Volterra-inspired system modelling GDP and FDI dynamics. The sensitivity analysis revealed that the system is highly responsive to changes in the interaction terms, as well as the quadratic growth terms for GDP and FDI. Understanding these sensitivities is critical for crafting policies that ensure sustainable economic growth without triggering excessive volatility. Future work could explore the inclusion of external economic factors, such as trade or inflation, to expand the model's applicability.

The model exhibits a moderate level of robustness, with varying sensitivities to changes in its coefficients. While it can tolerate certain adjustments in some coefficients without a significant loss in accuracy, it is highly sensitive to changes in others, particularly small increases or decreases in specific coefficients.

Coefficients a, c and f are less robust when increased, as even small upward changes lead to high sensitivity indices, meaning the model's accuracy is easily affected. These coefficients require careful management to prevent performance degradation.

Coefficients d and e show reduced robustness when decreased, with the model being particularly sensitive to even slight reductions. However, the model can handle larger increases in these coefficients more comfortably.

Coefficient b has a more balanced sensitivity to both increases and decreases, indicating a higher degree of robustness for moderate changes in this coefficient.

Overall, the model demonstrates a mix of robustness across its coefficients, being more vulnerable to increases in some and decreases in others. Maintaining the model's accuracy requires careful calibration of each coefficient within its allowable range.

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